

# A NOTE ON THE CONSTRUCTION OF FINITELY INJECTIVE MODULES

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ABSTRACT. We develop a technique to construct finitely injective modules which are non trivial, in the sense that they are not direct sums of injective modules. As a consequence, we prove that a ring  $R$  is left noetherian if and only if each finitely injective left  $R$ -module is trivial, thus answering an open question posed by Salce.

## 1. INTRODUCTION

It has been recently shown by Salce in [7] that if  $R$  is a non noetherian Matlis valuation domain, then there exists a nontrivial finitely injective  $R$ -module. Where an  $R$ -module  $M$  is called finitely injective if each finite subset of  $M$  is contained in an injective submodule (which is necessarily a direct summand). And a finitely injective module is said to be trivial if it is a direct sum of injective modules. This result was inspired by an older characterization of noetherian rings obtained by Ramamurthi and Rangaswamy in [6]. Namely, they proved that a ring  $R$  is left noetherian if and only if every finitely injective left  $R$ -module is injective. At this point, the natural question of whether general left noetherian rings can be characterised in terms of the existence of nontrivial finitely injective modules naturally arises (see [7, Question 1]).

The goal of this note is to give a positive answer to the above question, as well as provide a simpler and more natural tool to construct nontrivial finite injective modules over non noetherian rings. Using this new construction, we prove that an injective left  $R$ -module  $M$  is  $\Sigma$ -injective if and only if any finitely injective submodule of the injective envelope of  $M^{(\aleph_0)}$  is trivial. In particular, we deduce that a ring is left noetherian if and only if any finitely injective left module is a direct sum of injective modules.

Salce's construction of non-trivial finitely injective modules is based on a classical construction by Hill for Abelian groups (see [4]). This construction was later generalized by Griffith [3] and Huisgen-Zimmermann [8] for modules in order to characterize left perfect rings in terms of the existence of nontrivial flat and strict Mittag-Leffler (equivalently, locally projective) modules. We would like to stress that Hill's construction needs to be applied to countably generated modules (usually, projective modules). Indeed, Salce's example of a non-trivial finitely injective module is obtained by applying this Hill's construction to a countably generated injective module which is not  $\Sigma$ -injective. The existence of these countably generated modules is guaranteed over non noetherian Matlis valuation domains, but not over arbitrary rings. Therefore, we need to develop in this note a new (and much

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simpler) explicit construction of non trivial finitely injective modules that can be applied to any injective module which is not  $\Sigma$ -injective.

## 2. FINITELY INJECTIVE MODULES OVER NON-NOETHERIAN RINGS

Along this note,  $R$  will denote a ring with unit and module will mean a left  $R$ -module unless otherwise is stated. Morphisms will operate on the right and the composition of  $f : A \rightarrow B$  and  $g : B \rightarrow C$  will be denoted by  $fg$ . Given a left  $R$ -module  $M$ , we will denote by  $E(M)$  its injective hull.

Recall that an injective module  $M$  is said to be  $\Sigma$ -injective if any direct sum of copies of  $M$  is injective. Our main theorem characterizes  $\Sigma$ -injective modules in terms of finitely injective submodules of the injective hull of a countably direct sum of copies of the module.

**Theorem 2.1.** *Let  $M$  be an injective module. The following assertions are equivalent:*

- (1)  $M$  is  $\Sigma$ -injective.
- (2) Every finitely injective submodule of  $E(M^{(\aleph_0)})$  is injective.
- (3) Every finitely injective submodule of  $E(M^{(\aleph_0)})$  is a direct sum of injective modules.

**Proof.** 1)  $\Rightarrow$  2). Let us first note that  $E(M^{(\aleph_0)}) = M^{(\aleph_0)}$  is  $\Sigma$ -injective too and, consequently, it is a direct sum of indecomposable modules (see [1]). Now, using [5, Theorem 2.22] and [5, Lemma 2.16] we get that the union of any chain of direct summands of  $M^{(\aleph_0)}$  is a direct summand.

Let  $N$  be a finitely injective submodule of  $M^{(\aleph_0)}$  and  $\{x_\alpha \mid \alpha < \kappa\}$  be any generating set of  $N$  (where  $\kappa$  is an ordinal). We claim that  $N$  is the union of a chain  $\{N_\alpha \mid \alpha < \kappa\}$  of direct summands of  $M^{(\aleph_0)}$ . By the previous discussion, this implies that  $N$  is a direct summand of  $M^{(\aleph_0)}$  and, in particular, injective. We are going to construct this chain of submodules of  $N$  recursively on  $\alpha$  with the property that  $x_\alpha \in N_\alpha$  for each  $\alpha < \kappa$ .

For  $\alpha = 0$ , choose an injective submodule  $N_0$  of  $N$  that contains  $x_0$ . We know that this  $N_0$  does exist since we are assuming that  $N$  is finitely injective. As  $N_0$  is injective, it is a direct summand of  $M^{(\aleph_0)}$  and therefore, of  $N$ .

Let now  $\alpha < \kappa$  be any ordinal and assume that we have constructed our chain  $\{N_\gamma \mid \gamma < \alpha\}$  of submodules of  $N$ . Then note that  $\bigcup_{\gamma < \alpha} N_\gamma$  is a direct summand of  $M^{(\aleph_0)}$  (and therefore, of  $N$ ), since it is the union of a chain of direct summands of  $M$  and  $M$  is  $\Sigma$ -injective. Let us write  $N = \left(\bigcup_{\gamma < \alpha} N_\gamma\right) \oplus E$  for some  $E \leq F$ , and  $x_\alpha = n + e$  for some  $n \in \bigcup_{\gamma < \alpha} N_\gamma$  and  $e \in E$ . Since  $E$  is finitely injective, there exists an injective submodule  $L$  of  $E$  containing  $e$ . Let  $E(Re)$  be an injective hull of  $Re$  inside  $L$ . Then  $E(Re) \cap \left(\bigcup_{\gamma < \alpha} N_\gamma\right) = 0$  since  $Re$  is essential in  $E(Re)$  and  $Re \cap \left(\bigcup_{\gamma < \alpha} N_\gamma\right) = 0$ . Set then  $N_\alpha = \left(\bigcup_{\gamma < \alpha} N_\gamma\right) \oplus E(Re)$  which contains  $x_\alpha$  and, by [5, Proposition 2.2], it is a direct summand of  $M^{(\aleph_0)}$ . This concludes the construction.

2)  $\Rightarrow$  3). Trivial.

3)  $\Rightarrow$  1). Suppose that (1) is false. We are going to construct a finitely injective submodule of  $E(M^{(\aleph_0)})$  which is not a direct sum of injective modules. As we

are assuming that  $M$  is not  $\Sigma$ -injective, we know that  $M^{(\aleph_0)}$  is not injective by [2, Proposition 3]. Denote by  $N = M^{(\aleph_0)}$  and by  $E = E(M^{(\aleph_0)})$ . By Baer's criterium, there exists a left ideal  $I$  of  $R$  and a morphism  $f : I \rightarrow N$  that cannot be extended to  $R$ . Let  $\Omega$  be the set of all submodules  $L$  of  $E$  such that:

- (1)  $L$  is finitely injective;
- (2)  $N \leq L$ , and
- (3) The morphism  $f : I \rightarrow N$  cannot be extended to a morphism  $R \rightarrow L$ .

Clearly,  $\Omega$  is a non-empty partially ordered set. Let us show that it is inductive. Let  $\{L_k : k \in K\}$  be a chain in  $\Omega$  and let us prove that  $L = \bigcup_{k \in K} L_k \in \Omega$ . Trivially,  $L$  satisfies conditions (1) and (2). Suppose that there exists an extension  $g : R \rightarrow L$  of  $f$ . Then, as  $R$  is finitely generated, there exists a  $k \in K$  such that  $\text{Im } f \leq L_k$ . But this means that  $g$  is an extension of  $f$  over  $L_k$ , which contradicts that  $L_k \in \Omega$ . Thus,  $L$  also satisfies (3) and it is an element of  $\Omega$ .

Let now  $L$  be a maximal element of  $\Omega$ . We are going to show that  $L$  cannot be a direct sum of injective modules. Suppose on the contrary that  $L$  would be a direct sum of injective modules, say  $L = \bigoplus_{t \in T} E_t$ . Denote by  $q_t : L \rightarrow E_t$  the canonical projection, for each  $t \in T$ . Let

$$T' = \{t \in T : f q_t \neq 0\}$$

and note that  $T'$  is infinite because otherwise,  $\text{Im } f$  would be contained in a finite direct subsum of  $\bigoplus_{t \in T} E_t$  and  $f$  would have an extension to  $R$ . Call  $q_{T'} : L \rightarrow \bigoplus_{t \in T'} E_t$  the projection. And write  $T' = T_1 \cup T_2$  as a disjoint union of two infinite subsets. Denote by  $q_{T_i} : \bigoplus_{t \in T'} E_t \rightarrow \bigoplus_{t \in T_i} E_t$  the projections, for  $i = 1, 2$ .

We claim that neither  $f q_{T'} q_{T_1}$  nor  $f q_{T'} q_{T_2}$  can be extended to  $R$ . Assume on the contrary that, for instance,  $f q_{T'} q_{T_1}$  could be extended to a morphism  $h : R \rightarrow \bigoplus_{t \in T_1} E_t$ . Then there would exist a finite subset  $F \subseteq T_1$  such that  $\text{Im } h \subseteq \bigoplus_{t \in F} E_t$ , as  $\text{Im } h$  is finitely generated. But then  $0 = f q_{T'} q_{T_1} q_t = f q_t$  for every  $t \in T_1 \setminus F$ . A contradiction that proves our claim.

Let finally  $L' = E(\bigoplus_{t \in T_1} E_t) \oplus (\bigoplus_{t \in T \setminus T_1} E_t)$ . Then  $L'$  belongs to  $\Omega$ , since  $f$  does not have an extension to  $R$  because  $T_2 \subseteq T \setminus T_1$  and  $f q_{T'} q_{T_2}$  cannot be extended to  $R$ . Moreover,  $L$  is strictly contained in  $L'$ , as  $\bigoplus_{t \in T_1} E_t$  is not injective. But this contradicts the maximality of  $L$ .  $\square$

**Remark 2.2.** *Let us note that our proof of (3)  $\Rightarrow$  (1) in the above theorem gives a general tool to construct finitely injective modules which cannot be injective. This construction is simpler than the one obtained [7] and seems more natural on this injectivity context.*

As a consequence, we get a possitive answer to [7, Question 1].

**Corollary 2.3.** *Let  $R$  be a ring. The following assertions are equivalent:*

- (1)  $R$  is left noetherian.
- (2) Each finitely injective left  $R$ -module is injective.
- (3) Each finitely injective left  $R$ -module is a direct sum of injective modules.

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